Linear orbital stability of discrete shock profiles for systems of conservation laws

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- **Context and definition of discrete shock profiles**
- **Existence results**
- **Stability of discrete shock profiles**
  - Definition of the nonlinear orbital stability and overview of results
  - Main result : Spectral stability implies linear orbital stability

Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{gathered}\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \end{gathered} \tag{1}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

The result that will be presented also holds for systems of conservations laws.

This type of PDE tends to have solutions with discontinuities.

Larger goal: We want to know if numerical schemes obtained by discretizing (1) can approach correctly those discontinuous solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ and a speed $s \in \mathbb{R}$. The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

is a weak solution of the scalar conservation law if and only if

$$f(u^-) - f(u^+) = s(u^- - u^+). \quad \text{(Rankine-Hugoniot condition)}$$

It is a Lax shock when

$$f'(u^+) < s < f'(u^-).$$

The main result of the presentation will focus on steady Lax shocks, i.e. when $s = 0$.

Conservative finite difference schemes

We consider a conservative explicit finite difference scheme

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N} u^n$$

where :

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$ is the initial condition.
- The nonlinear discrete evolution operator $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is defined for $u = (u_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \dots, u_{j+q-1}\right)\right).$$

- The numerical flux $F :]0, +\infty[\times \mathbb{R}^{p+q} \to \mathbb{R}^d$ is a smooth function.
- The integers $p, q \in \mathbb{N} \setminus \{0\}$ give the size of the stencil of the scheme.
- We fix $\nu = \frac{\Delta t}{\Delta x} > 0$ the ratio between the time and space steps.

Assumptions:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \ldots, u) = f(u)$ (consistency condition)
- We choose ν which must satisfy a CFL condition

$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p$$

  for some neighborhood $\mathcal{U}$ of the states $u^{\pm}$.
- We assume to have linear-ℓ^2 stability of the scheme about all the states $u \in \mathcal{U}$.
- The scheme introduces numerical viscosity. In the present presentation, we consider a first order scheme. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$. For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n)$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

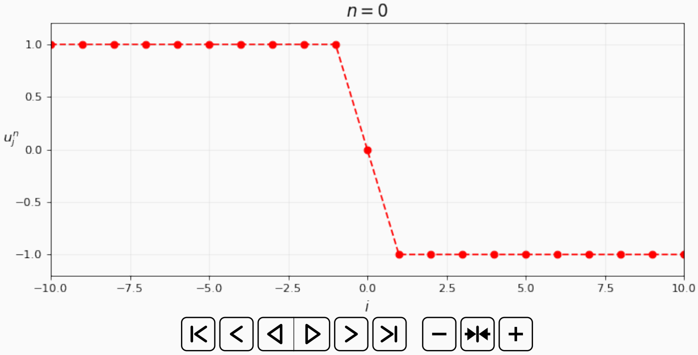
$$\overline{u}(x) \underset{x \to \pm\infty}{\to} u^{\pm}.$$

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j \to \pm\infty}{\to} u^{\pm}.$$

Example : We consider the initial condition (mean of the standing shock on each cell $\left[(j-\frac{1}{2})\Delta x,(j+\frac{1}{2})\Delta x\right[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



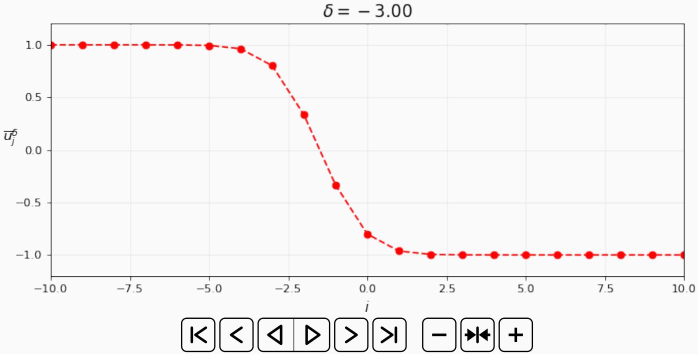
A desirable feature of the numerical scheme should be that stable shock waves for the conservation laws should yield stable DSPs for the numerical scheme. This separates the theory surrounding DSPs in two parts:

Existence of DSPs   Stability of DSPs

From now on we will focus on elements of theory surrounding stationary discrete shock profiles ($s=0$).

Existence results on SDSPs

Example : We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.



For standing Lax shocks, one aims to have the existence of a differentiable one-parameter family $(\overline{u}^{\delta})_{\delta \in]-\varepsilon,\varepsilon[}$ of SDSPs.

- Jennings, *Discrete shocks* (1974)
    - scalar case
    - for shocks satisfying Oleinik's E-condition
    - conservative monotone scheme
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* (1979)
    - system case
    - weak Lax shocks
    - first order scheme
- Michelson, *Discrete shocks for difference approximations to systems of conservation laws* (1984)
    - extension of Majda-Ralston for third order scheme
- Different cases: Smyrlis (1990), Liu-Yu (1999), Serre (2004) etc...

Stability of discrete shock profiles

The end goal would be to prove a property of nonlinear orbital stability for the DSPs:

For small admissible perturbations h, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + h$ converges towards the set of translations of the DSP $\left\{\overline{u}^\delta, \delta \in]-\varepsilon, \varepsilon[\right\}$.

We are going to present a possible first step towards a quite general result of nonlinear orbital stability.

Known stability results

- Jennings, *Discrete shocks* (1974)
  - scalar case
  - conservative monotone scheme
  - nonlinear orbital stability for ℓ^1 perturbations
- Liu-Xin, *L^1-stability of stationary discrete shocks*, (1993)
  - system case
  - Lax-Friedrichs scheme
  - weak Lax shocks
  - zero mass perturbation (dropped in Ying (1997))
- Michelson, *Stability of discrete shocks for difference approximations to systems of conservation laws*, (2002)
  - system case
  - weak Lax shocks
  - First and third order schemes
- Different cases: Smyrlis (1990), Liu-Yu (1999), etc...

One would hope to prove a result of nonlinear orbital stabilty in the system case, for a fairly large class of numerical schemes and with no smallness assumption on the amplitude of the shock.

The first idea

Our first goal is to study the semigroup associated to the operator $\mathcal{L}$ obtained by linearizing $\mathcal{N}$ about the SDSP $\overline{u}$.

We introduce a zero mass perturbation $h^0 \in \ell^1(\mathbb{Z})$. We then define

$$v^0 = \overline{u} + h^0$$

and

$$\forall n \in \mathbb{N}, v^{n+1} = \mathcal{N}(v^n). \tag{2}$$

If we define $h^n = v^n - \overline{u}$, then (2) yields

$$h^{n+1} = \mathcal{L}h^n + Q(h^n, \overline{u})$$

with $Q(h^n, \overline{u})$ being some "quadratic" term. Duhamel's formula implies that a precise understanding of the behavior of the family of operators $(\mathcal{L}^n)_{n\geq 0}$ is necessary at this point.

The second idea

We want to study the Green's function associated to the operator $\mathcal{L}$ using the techniques developed in Zumbrun-Howard, *Pointwise semigroup methods and stability of viscous shock waves* (1998) to study traveling waves for parabolic PDEs.

Extension of the result of Lafitte-Godillon, *Green's function pointwise estimates for the modified Lax-Friedrichs scheme*, (2003)

Linearization of the numerical scheme about the constant states $u^\pm$

We define the bounded operator $\mathcal{L}^\pm : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about the constant state $u^\pm$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}^\pm h)_j := \sum_{k=-p}^{q} a_k^\pm h_{j+k}.$$

This is a Laurent operator/convolution operator. Its spectrum is given by

$$\sigma(\mathcal{L}^\pm) = \left\{ \sum_{k=-p}^{q} a_k^\pm e^{itk}, t \in \mathbb{R} \right\}.$$

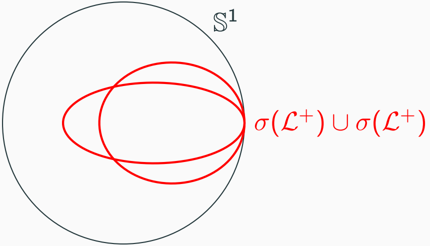
We assume that

$$\forall \kappa \in \mathbb{S}^1 \setminus \{1\}, \quad \left| \sum_{k=-p}^{q} \kappa^k a_k^{\pm} \right| < 1 \quad (\ell^2 - \text{stability})$$

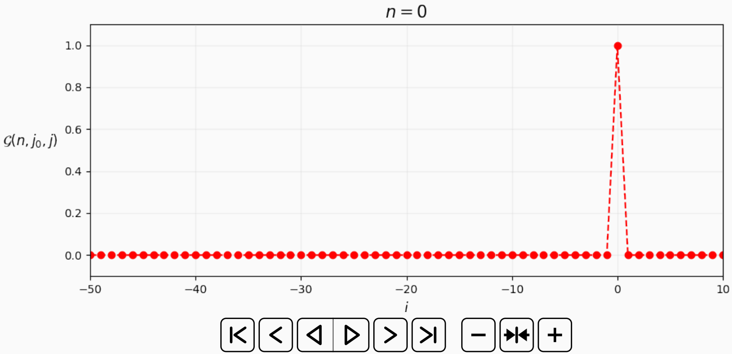
and that there exists a complex number $\beta_\pm$ with positive real part such that

$$\sum_{k=-p}^{q} a_k^{\pm} e^{itk} \underset{t \to 0}{=} \exp(-if'(u^{\pm})\nu t - \beta_\pm t^2 + O(|t|^3)). \quad \text{(Diffusivity condition)}$$

(see fondamental contribution of [14])



Green's function associated to the operator $\mathcal{L}^+$



The Gaussian behavior has been studied thoroughly in recent extensions on the local limit theorem (see [3, 12, 2, 1]).

Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$. We are interested in solutions of the linearized numerical scheme

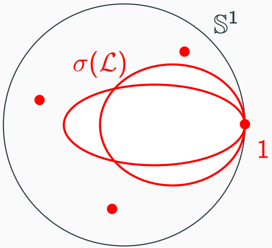
$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L} h^n, \qquad h^0 \in \ell^2(\mathbb{Z}).$$

We define the Green's function

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Observation on the spectrum of $\mathcal{L}$

The elements of the unbounded component of $\mathbb{C}\backslash\sigma(\mathcal{L}^+)\cup\sigma(\mathcal{L}^-)$ are either eigenvalues of $\mathcal{L}$ or are in its resolvent set.



Spectral stability assumption

- In the article, we construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$.

  "$\mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta$ and thus $\mathcal{L}\frac{\partial \overline{u}^\delta}{\partial \delta} = \frac{\partial \overline{u}^\delta}{\partial \delta}$."

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.

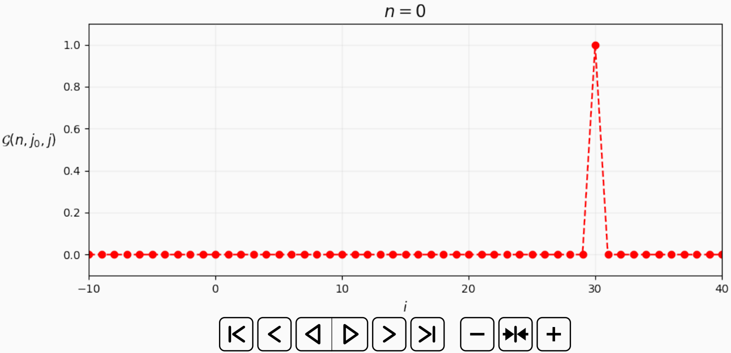
Theorem
Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j - j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}
$$

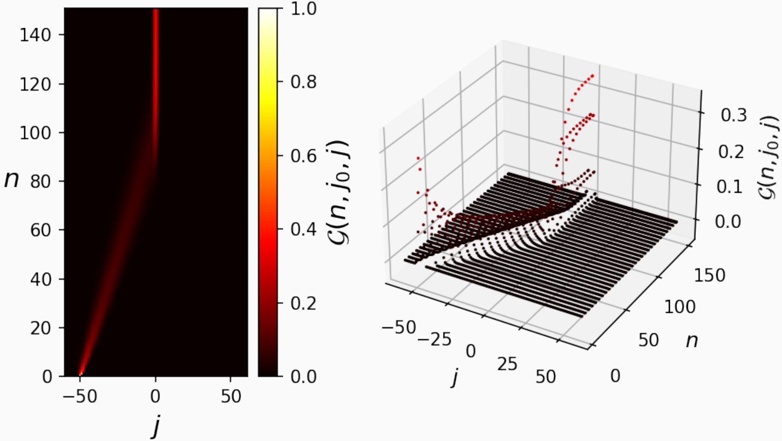
where $E(x) \underset{x\to-\infty}{\to} 1$ and $E(x) \underset{x\to+\infty}{\to} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$

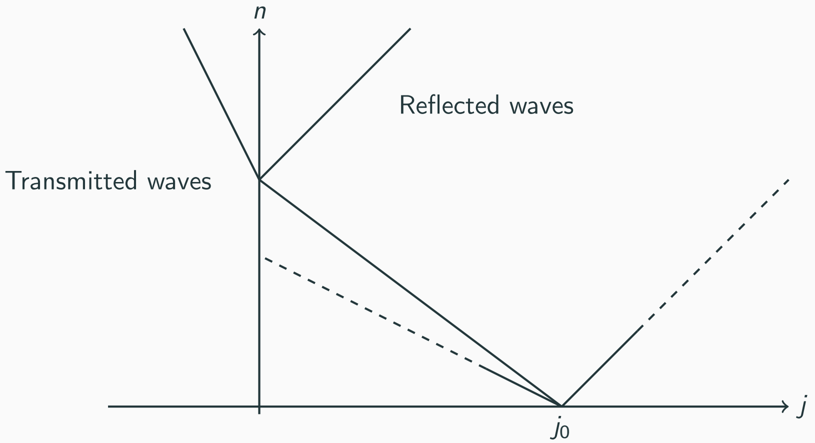


$j_0 = -50$



140
120
100
80
60
40
20
0
n
−50
0
50
j
1.0
0.8
0.6
0.4
0.2
0.0
$\mathcal{G}(n, j_0, j)$
0.3
0.2
0.1
0.0
$\mathcal{G}(n, j_0, j)$
−50
−25
0
25
50
j
150
100
50
0
n

Case of systems



• Using the inverse Laplace tranform with Γ a path that surrounds the spectrum $\sigma(\mathcal{L})$, we have

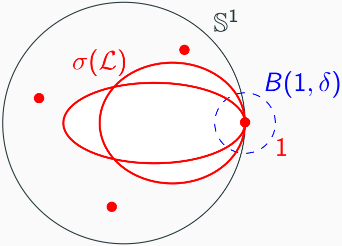
$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_\Gamma z^n \left((zId - \mathcal{L})^{-1} \delta_{j_0}\right)_j dz. \tag{3}$$

• We rewrite the eigenvalue problem

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z) W_j. \tag{4}$$



We are interested in solutions of (4) that tend towards 0 as j tends to $+\infty$ or $-\infty$ (Jost solutions, geometric dichotomy) and use them to express find an expression and meromorphically extend $z \mapsto \left((zId - \mathcal{L})^{-1} \delta_{j_0}\right)_j$ through the essential spectrum near 1.

• Using this idea and a good choice of path Γ, we prove sharp estimates on the temporal Green's function.

Theorem
Under the same assumption as for the previous theorem, for $p \in [1, +\infty]$, there exists a positive constant C such that

$$\forall h \in \ell^1(\mathbb{Z}, \mathbb{C}^d), \forall n \in \mathbb{N}, \qquad \min_{V \in \ker(Id_{\ell^2} - \mathcal{L})} \|\mathcal{L}^n h - V\|_{\ell^p} \leq \frac{C}{n^{\frac{1}{2}\left(1 - \frac{1}{p}\right)}} \|h\|_{\ell^1}.$$

Conclusion/ Perspective / Open questions

About the theorem:

- Bounds uniform in j_0
- Very few limitation on the size of the stencil
- The result can be proved for systems
- The result can be proved for higher odd ordered schemes (not only for first order schemes)

Perspective:

- Can we now prove nonlinear orbital stability ? (at least in the scalar case?)
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed)?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws (Carbuncle phenomenon)

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